Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.4 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 9 in “Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)”.

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Counting

9.6 r-Combinations with Repetition Allowed

In this lecture:

- ❑ Part 1:
- ❑ Part 2:

What is this section about?

In this chapter we discussed how to count the numbers of **ways of choosing *k* elements from *n***

	Strings: Order Matters	Sets: Order Doesn't Matter
Repetition Allowed	n^k **Select 4-digites PIN, from 0-9** $=10.10.10.10 = 10^4 = 10000$	? **Select 10 cans, from 4 types of drinks** =?
Repetition not Allowed	$P(n, k)$ **Select 4-digites PIN, from 0-9** =10!/6!=10.9.8.7 = 5040	$\binom{n}{k}$ **Select 4-position team, from 10 people** =10!/4!.6! = 5040/24 = 210

r-Combinations with Repetition Allowed

Examples:
- buy 20 drinks of cola, 7up, or fanta. How many ways?
- select a committee of 3 people, from 10 persons, but one person may play one or more roles.

Given a set on n elements $\{x_1, x_2, \ldots, x_n\}$

Choose r element $[x_{i1}, x_{i2}, \ldots x_{ik}]$

With **repetition** allowed, and **unordered**.

• Definition

An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

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Example

Find the number of 3-combinations with repetition allowed, or **multisets** of size 3, that can be selected from {1, 2, 3, 4}

[1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]
[1, 2, 2] ; [1, 2, 3]; [1, 2, 4];
[1, 3, 3] ; [1, 3, 4]; [1, 4, 4];
[2, 2, 2] ; [2, 2, 3]; [2, 2, 4];
[2, 3, 3] ; [2, 3, 4]; [2, 4, 4];
[3, 3, 3] ; [3, 3, 4]; [3, 4, 4];
[4, 4, 4]

20 ways

➔ How to calculate this automatically?

→ Can to "see" this multiset problem as a string problem?

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Calculating r-Combinations with Repetition Allowed

Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed.

Category 1	Category 2	Category 3	Category 4	Result of the Selection
	×		× ×	1 from category 2 2 from category 4
×		×	×	1 each from categories 1, 3, and 4
× × ×				3 from category 1

× × | | × | means [1,1,3]

The problem now became like selecting 3 positions out of 6, because once 3 positions have been chosen for the ×'s, the |'s are placed in the remaining 3 positions, which is $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{3!}} = 20$

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Calculating r-Combinations with Repetition Allowed

Category 1	Category 2	Category 3	. . .	Category $n-1$	Category n
|	|	|	. . . |	|	

r ×'s to be placed in categories

$n-1$ vertical bars (to separate the n categories)
r crosses (to represent the r elements to be chosen).

r-combanations of xs and |s

Given $(r+n-1)$ select r $\binom{r+n-1}{r}$

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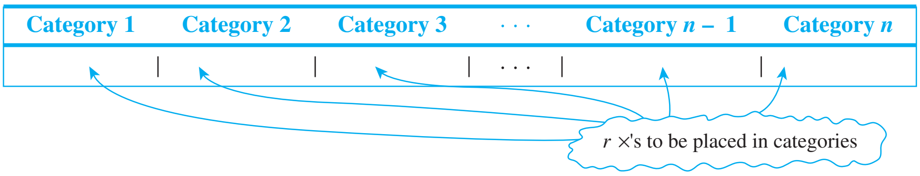
Calculating r-Combinations with Repetition Allowed



n – 1 vertical bars (to separate the n categories)
r crosses (to represent the r elements to be chosen).

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

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Exercise 1.a

A person giving a party wants to buy 15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

How many different selections of cans of 15 soft drinks can he make?

Can be represented by a string of 5 – 1 = 4 vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected). For instance,

$$\times\times\times \mid \times\times\times\times\times\times\times \mid \quad \mid \times\times\times \mid \times\times$$

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19\cdot \overset{6}{\cancel{18}}\cdot 17\cdot \overset{2}{\cancel{16}}\cdot \cancel{15!}}{\cancel{15!}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 3{,}876.$$

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Exercise 1.b

A person giving a party wants to buy15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

If root beer is one of the types of soft drink, how many different selections include at least 6 cans of root beer?

Thus we need to select 9 cans from the 5 types.
The nine additional cans can be represented as 9 ×'s and 4 |'s.

$$\binom{9+4}{9} = \binom{13}{9} = \frac{13 \cdot \cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{9!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 715.$$

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Exercise 2

Counting Triples (i, j, k) with $1 \leq i \leq j \leq k \leq n$

If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct? In other words, how many triples of integers (i, j, k) are there with $1 \leq i \leq j \leq k \leq n$?

Category 1		2		3		4		5	Result of the Selection
	|		|	× ×	|		|	×	(3, 3, 5)
×	|	×	|		|	×	|		(1, 2, 4)

$$\binom{3+(n-1)}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!(n+2-3)!}$$

$$= \frac{(n+2)(n+1)n\cancel{(n-1)!}}{3!\cancel{(n-1)!}} = \frac{n(n+1)(n+2)}{6}.$$

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Exercise

Counting Iterations of a Loop

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run?

for $k := 1$ **to** n
  for $j := 1$ **to** k
    for $i := 1$ **to** j
      [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]
    next i
  next j
next k

$$\binom{3+(n-1)}{3} = \frac{n(n+1)(n+2)}{6}$$

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Exercise

The Number of Integral Solutions of an Equation

How many solutions are there to the equation $x1 + x2 + x3 + x4 = 10$ if $x1, x2, x3$, and $x4$ are nonnegative integers?

Categories x_1	x_2	x_3	x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × |	× × × × × |	|	× × ×	$x_1 = 2, \quad x_2 = 5, \quad x_3 = 0, \quad \text{and} \quad x_4 = 3$
× × × × |	× × × × × × |	|		$x_1 = 4, \quad x_2 = 6, \quad x_3 = 0, \quad \text{and} \quad x_4 = 0$

$$\binom{10+3}{10} = \binom{13}{10} = \frac{13!}{10!(13-10)!} = \frac{13\cdot 12\cdot 11\cdot \cancel{10!}}{\cancel{10!}\cdot 3\cdot 2\cdot 1} = 286.$$

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Exercise

Additional Constraints on the Number of Solutions

How many integer solutions are there to the equation $x1 + x2 + x3 + x4 = 10$ if each $x_i \geq 1$?

$$\binom{6+3}{6} = \binom{9}{6} = \frac{9!}{6!(9-6)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1} = 84.$$

Start by putting one cross in each of the four categories, then distribute the remaining six crosses among the categories

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Summary

In this chapter we discussed how to count the numbers of **ways of choosing *k* elements from *n***

	Order Matters	Order Doesn't Matter
Repetition Allowed	n^k	$\binom{k+n-1}{k}$
Repetition not Allowed	$P(n,k)$	$\binom{n}{k}$

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